

Note

Boolean inequations[☆]

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Abstract

In this paper we consider Boolean inequations i.e. the inequations of the form $f(X) \neq 0$, where f is a Boolean function. The basic idea in this paper is: the inequation $f(X) \neq 0$ means that there exists p such that $f(X) = p$ and $p \neq 0$. We give the formula which determines all the solutions of Boolean inequation.

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Many authors considered Boolean equations. Löwenheim [3,4] made a great contribution to the research of general and reproductive general solutions of Boolean equations. The basic facts of Boolean equations and various forms of the solutions of Boolean equations can be found in Rudeanu's monograph [5]. Banković [1,2] gave some methods for solving Boolean equations. The results published after 1974 were presented in [6, Chapter 6].

S. Rudeanu raised [5, Problem 10.1] the following problem: develop the theory of Boolean inequalities and that of alternative systems of Boolean equations. In this paper we deal with a part of this theory.

Let $(B, \cap, \cup, ', 0, 1)$ be a Boolean algebra and n be a natural number.

Definition 1. Let $x \in B$. Then

$$x^1 = x, \quad x^0 = x'.$$

If $X = (x_1, \dots, x_n) \in B^n$ and $A = (a_1, \dots, a_n) \in \{0, 1\}^n$ then

$$X^A = x_1^{a_1} \cap \dots \cap x_n^{a_n}.$$

In the sequel \cap will be omitted.

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Theorem 1 (Rudeanu [5], Corollary 1). *The function $f : B^n \rightarrow B$ is Boolean if and only if it can be written in the canonical disjunctive form*

$$f(X) = \bigcup_A f(A)X^A.$$

Let B be a Boolean algebra. For every $t \in B$, $V = (v_1, \dots, v_n) \in B^n$, we put

$$Vt = tV = (tv_1, \dots, tv_n).$$

Lemma 1 (Rudeanu [5], Lemma 2.3). *Let $f : B^n \rightarrow B$ be a Boolean function. For every $t \in B$, $V, W \in B^n$, the equality*

$$f(tV \cup t'W) = tf(V) \cup t'f(W)$$

holds.

1. Boolean equations

To solve Boolean equation $f(X) = 0$ means to determine all $X \in B^n$ such that $f(X) = 0$ holds i.e. to determine the set $S = \{X | f(X) = 0 \wedge X \in B^n\}$. Note that

$$f(X) = 0 \Leftrightarrow X \in S.$$

Theorem 2 (Rudeanu [5], Theorem 2). *Let $f : B^n \rightarrow B$ be a Boolean function. The equation $f(X) = 0$ is consistent (has a solution) if and only if*

$$\prod_A f(A) = 0.$$

Let $T = (t_1, \dots, t_n)$.

Definition 2. Let $f, F_1, \dots, F_n : B^n \rightarrow B$ be Boolean functions and $F = (F_1, \dots, F_n)$. The formula

$$X = F(T),$$

or in scalar form

$$x_i = F_i(t_1, \dots, t_n), \quad (i = 1, \dots, n)$$

defines a general solution of the consistent Boolean equation $f(X) = 0$ if and only if, for every $X \in B^n$,

$$f(F(X)) = 0 \wedge (f(X) = 0 \Rightarrow (\exists T)X = F(T)).$$

One can prove that the previous formula is equivalent to

$$f(X) = 0 \Leftrightarrow (\exists T)X = F(T).$$

Lemma 2 (Rudeanu [5], Lemma 2.3). *Assume that the equation*

$$cx \cup dx' = 0$$

is consistent. Then

$$cx \cup dx' = 0 \Leftrightarrow (\exists t)(x = c't \cup dt') \tag{1}$$

for all $x \in B$.

Theorem 3 (Rudeanu [5], Theorem 2.11). Let $f : B^n \rightarrow B$ be a Boolean function. If $f(Q) = 0$ then

$$f(X) = 0 \Leftrightarrow (\exists T) X = f'(T)T \cup f(T)Q$$

for all $X \in B^n$.

Let $k = 2^n - 1$.

Theorem 4 (Banković [1]). Let $f : B^n \rightarrow B$ be a Boolean function. If $f(X) = 0$ is consistent then, for every $X \in B^n$,

$$f(X) = 0 \Leftrightarrow (\exists T) X = \bigcup_{i=0}^k (f'(A_i)A_i \cup f(A_i)f'(A_{i_1})A_{i_1} \cup f(A_i)f(A_{i_1})f'(A_{i_2})A_{i_2} \cup \dots \cup f(A_i)f(A_{i_1})f(A_{i_2}) \dots f(A_{i_{p-1}})f'(A_{i_p})A_{i_p})T^{A_i},$$

where for every $i \in \{0, 1, \dots, k\}$ $(A_i, A_{i_1}, \dots, A_{i_p})$ is a permutation of $\{0, 1\}^n$.

2. Boolean inequations

Let $f : B^n \rightarrow B$ be a Boolean function. The relation

$$f(X) \neq 0$$

is called a Boolean inequation.

To solve Boolean inequation $f(X) \neq 0$ means to determine all $X \in B^n$ such that $f(X) \neq 0$ holds.

Theorem 5 (Rudeanu [5], Remark 10.5). Let $f : B^n \rightarrow B$ be a Boolean function. Inequation $f(X) \neq 0$ has a solution if and only if $\bigcup_A f(A) \neq 0$.

Theorem 6. Let $f : B^n \rightarrow B$ be a Boolean function. If $f(P) \neq 0$ and

$$X = f(T)T \cup Pf'(T) \tag{2}$$

then $f(X) \neq 0$ for all $T \in B^n$.

Proof. In accordance with Lemma 1, we have

$$f(Tf(T) \cup Pf'(T)) = f(T)f(T) \cup f(P)f'(T) = f(T) \cup f(P) \neq 0. \quad \square$$

Example. Let $B = \{0, 1, \alpha, \alpha'\}$. Consider the following Boolean inequation:

$$\alpha x \cup \alpha' x' \neq 0. \tag{3}$$

Since 0 satisfies (3), Theorem 6 yields the solution

$$x = t(\alpha t \cup \alpha' t') \cup 0 \cdot (\alpha t \cup \alpha' t')' = \alpha t.$$

Taking $t \in \{0, 1, \alpha, \alpha'\}$ formula $x = \alpha t$ gives $x = 0$ or $x = \alpha$. Since $\alpha x \cup \alpha' x' = 0$ if and only if $x = \alpha'$ it follows that the set of all the solutions of inequation (3) is $\{0, 1, \alpha\}$. So, the formula $x = \alpha t$, obtained by (2), does not determine all the solutions of inequation (3). One can prove that there is no Boolean function $g : B \rightarrow B$ having the range $\{0, 1, \alpha\}$.

To describe all the solutions of an inequation $f(x) \neq 0$, where $f : B \rightarrow B$ is a Boolean function, we shall use a Boolean function with two variables.

We are going to use the ring sum $u + v = u'v \cup uv'$, where $u, v \in B$.

Lemma 3. Let $f, g : B^n \rightarrow B$ be Boolean functions, Then

$$\bigcup_A f(A)X^A + \bigcup_A g(A)X^A = \bigcup_A (f(A) + g(A))X^A.$$

Proof. Using the following equalities [5, Theorem 1.5]

$$\left(\bigcup_A c_A X^A\right)' = \bigcup_A c'_A X^A \quad \text{and} \quad \left(\bigcup_A c_A X^A\right) \left(\bigcup_A b_A X^A\right) = \bigcup_A c_A b_A X^A$$

we have

$$\begin{aligned} & \bigcup_A f(A)X^A + \bigcup_A g(A)X^A \\ &= \left(\bigcup_A f(A)X^A\right)' \bigcup_A g(A)X^A \cup \bigcup_A f(A)X^A \left(\bigcup_A g(A)X^A\right)' \\ &= \bigcup_A f'(A)X^A \bigcup_A g(A)X^A \cup \bigcup_A f(A)X^A \bigcup_A g'(A)X^A \\ &= \bigcup_A (f'(A)g(A) \cup f(A)g'(A))X^A \\ &= \bigcup_A (f(A) + g(A))X^A. \quad \square \end{aligned}$$

Theorem 7. Let $f : B^n \rightarrow B$ be a Boolean function. If $f(X) \neq 0$ is consistent then, for every $X \in B^n$,

$$f(X) \neq 0 \Leftrightarrow (\exists p)(\exists T)(X = \Phi(T, p) \wedge p \neq 0)$$

where $X = \Phi(T, p)$ expresses the general solution of the equation

$$\bigcup_A (f(A) + p)X^A = 0$$

in the unknowns X .

Proof. $f(X) \neq 0$ means that there exists p such that $f(X) = p$ and $p \neq 0$. Note that $f(X) = p \Leftrightarrow f(X) + p = 0$ [5, Remark 1.17]. Thus, we have

$$\begin{aligned} f(X) \neq 0 &\Leftrightarrow (\exists p)(f(X) = p \wedge p \neq 0) \\ &\Leftrightarrow (\exists p)(f(X) + p = 0 \wedge p \neq 0) \\ &\Leftrightarrow (\exists p) \left(\bigcup_A f(A)X^A + p \bigcup_A X^A = 0 \wedge p \neq 0 \right) \quad \left(\bigcup_A X^A = 1 \right) \\ &\Leftrightarrow (\exists p) \left(\bigcup_A f(A)X^A + \bigcup_A pX^A = 0 \wedge p \neq 0 \right) \\ &\Leftrightarrow (\exists p) \left(\bigcup_A (f(A) + p)X^A = 0 \wedge p \neq 0 \right) \quad (\text{Lemma 3}). \end{aligned}$$

Since $\bigcup_A (f(A) + p)X^A = 0$ is a Boolean equation, we have

$$\bigcup_A (f(A) + p)X^A = 0 \Leftrightarrow (\exists T)X = \Phi(T, p)$$

where $X = \Phi(T, p)$ represents the general solution of $\bigcup_A (f(A) + p)X^A = 0$. Therefore

$$f(X) \neq 0 \Leftrightarrow (\exists p)((\exists T)X = \Phi(T, p) \wedge p \neq 0),$$

which coincides with desired result. \square

Corollary 1. Let $f : B^n \rightarrow B$ be a Boolean function. If $f(X) \neq 0$ is consistent then, for every $X \in B^n$,

$$f(X) \neq 0 \Leftrightarrow (\exists p)(\exists T)(X = \Phi(T, p) \wedge p \neq 0)$$

where

$$\begin{aligned} \Phi(T, p) = & \bigcup_{i=0}^k ((f(A_i) + p)'A_i \cup (f(A_i) + p)(f(A_{i_1}) + p)'A_{i_1} \\ & \cup (f(A_i) + p)(f(A_{i_1}) + p)(f(A_{i_2}) + p)'A_{i_2} \\ & \cup \dots \cup (f(A_i) + p)(f(A_{i_1}) + p)(f(A_{i_2}) + p) \\ & \cup \dots \cup (f(A_{i_{k-1}}) + p)(f(A_{i_k}) + p)'A_{i_k})T^{A_i} \end{aligned}$$

and for every $i \in \{0, 1, \dots, k\}$ $(A_i, A_{i_1}, \dots, A_{i_p})$ is a permutation of $\{0, 1\}^n$.

Proof. From Theorems 7 and 4. \square

If we take $n = 1$, we get

Corollary 2. Let $a \cup b \neq 0$. Then, for every $x \in B$,

$$ax \cup bx' \neq 0 \Leftrightarrow (\exists p)(\exists t)(x = (a + p)'t \cup (b + p)t' \wedge p \neq 0).$$

Lemma 4 (Rudeanu [5], Theorem 8.5). A Boolean function $f : B \rightarrow B$ is surjective if and only if it is of the form

$$f(x) = a + x. \quad (4)$$

Theorem 8. The function $g : B^2 \rightarrow B$ defined by

$$g(t, p) = (a + p)'t \cup (b + p)t'$$

is surjective.

Proof. Let $t = a$. Then we have

$$\begin{aligned} g(p, a) &= (a + p)'a \cup (b + p)a' = (a + p + 1)a + (b + p)(a + 1) \\ &= (a + 1)a + pa + b(a + 1) + p(a + 1) = p + b(a + 1) = p + ba'. \end{aligned}$$

Since $g(p, a) = a'b + p$ is of the form (4), function g is surjective. \square

Theorem 8 shows that the values of the function g “cover” the set B . Some elements of B are the solutions of the inequation $ax \cup bx' \neq 0$, while the rest of them are the solutions of the equation $ax \cup bx' = 0$. Formula

$$x = (a + p)'a \cup (b + p)a'$$

determines all the solutions of Boolean inequation $ax \cup bx' \neq 0$ and all the solutions of Boolean equation $ax \cup bx' = 0$. Namely, if we take $p = 0$ we get $x = a't \cup bt'$, i.e. we get the formula which determines all the solutions of equation $ax \cup bx' = 0$, by Lemma 2. For $p \neq 0$, we get the formula which determines all the solutions of inequation $ax \cup bx' \neq 0$, by Corollary 2.

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